2 The linear programming problem

2.1 General formulation

Linear programming problem (LPP) is the problem of finding an extremum (maximum or minimum) of a linear function of several variables subject to linear constraints on these variables.

Example

Find the maximum value of the function

$$f(x_1; x_2) = 2x_1 + 5x_2 \to \max$$

under the following constraints on the variables x_1 and x_2

$$\begin{cases} x_1 + x_2 \leq 127 \\ 7x_1 - x_2 \leq 83 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

Linear function is called the target or objective function.

Constraints may not be as strict, i.e. in the form of strict or non-strict inequalities, and can be strict – in the form of equalities.

LPP are excellent mathematical models for a large number of economic (and other) problems, such as production planning, resource consumption, transportation and others.

2.2 The mathematical formulation of the problem

In general, the problem can be formally represented as follows (for the maximization problem):

$$\begin{aligned} &(1) : \max z = \max(c_1 x_1 + \cdots + c_n x_n) \\ &(2) : a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i, i = \overline{1, m_1} \\ &(3) : a_{i1}x_1 + \cdots + a_{in}x_n = bi, i = \overline{m_1, m} \\ &(4) : x_j \geq 0, j = \overline{1, n_1}, \\ &(5) : x_j \geq\leq 0, j = \overline{n_1, n}. \end{aligned}$$

Similarly, we can write a general formulation of the minimization problem.

However, such problems are solved when they are written using one of the specific forms. For the maximization problem, there are 4 types of special LP problems: *standard* and *canonical* forms.

2.2.1 Standard form of maximization LP problem

$$(1) : \max z = \max(c_1x_1 + \cdots + c_nx_n)$$
$$(2) : a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i, i = \overline{1,m}$$
$$(3) : x_j \geq 0, j = \overline{1,n}.$$

In matrix and vector form, this problem can be written as follows:

$$\begin{array}{lcl} \max z = \max CX & & \max z = \max CX \\ AX \leq B & \iff & A_iX \leq b_1, i = \overline{1,m} \\ X \geq 0 & & X \geq 0 \end{array}$$

2.2.2 Canonical form of maximization LP problem

$$(1) : \max z = \max(c_1x_1 + \cdots + c_nx_n)$$
$$(2) : a_{i1}x_1 + \cdots + a_{in}x_n = b_i, i = \overline{1,m}$$
$$(3) : x_j \geq 0, j = \overline{1,n}.$$

If a mathematical model of a linear programming problem has this form, then we say that the problem is presented in *canonical form*.

In matrix and vector form, this problem can be written as

$$\begin{array}{lcl} \max z = \max CX & & \max z = \max CX \\ AX = B & \iff & A_iX = b_1, i = \overline{1,m} \\ X \geq 0 & & X \geq 0 \end{array}$$

Any linear programming problem can be reduced to a linear programming problem in canonical form. To do this, in the general case, we must be able to reduce the problem of maximization to the minimization problem; move from the constraints of inequalities to equality constraints and replace variables that do not satisfy the condition of non-negativity. Maximization of a function equivalent to minimization of the same function taken with the opposite sign, and vice versa.

The rule of reduction of a linear programming problem to the canonical form is as follows:

- if the original problem requires to determine the maximum of a linear function, it is necessary to change the sign and search the minimum of this function;
- if the right side of constraint is negative, then multiply this constraint by -1;
- if there are constraints of inequality, by including additional non-negative variables are converted to equality;
- if some variable x_j has no restrictions on the sign, it will be replaced (in the objective function and all constraints) by the difference between the two new non-negative variables: $x_3 = x_3^+ - x_3^-,\ \ x_3^+, x_3^- \geq 0$.

Example 1. Reduction to the canonical form of the linear programming problem:

$$\begin{aligned}
&\min z = 2x_1 + x_2 - x_3\\
&2x_2 - x_3 \leq 5;\\
&x_1 + x_2 - x_3 \geq -1;\\
&2x_1 - x_2 \leq -3;\\
&x_1 \geq 0; x_2 \geq 0; x_3 \geq 0.
\end{aligned}$$

We introduce into each equation of constraints leveling variables x_4, x_5, x_6. The system takes the form of equations, and in the first and third equations of system constraints variables x_4, x_6 are introduced into left side with the sign "+", and into the second equation the variable x_5 is introduced with sign "–".

$$\begin{aligned}
&2x_2 - x_3 + \boldsymbol{x_4} = 5;\\
&x_1 + x_2 - x_3 - \boldsymbol{x_5} = -1;\\
&2x_1 - x_2 + \boldsymbol{x_6} = -3;\\
&x_4 \geq 0; x_5 \geq 0; x_6 \geq 0.
\end{aligned}$$

Free terms in canonical form must be positive, for this last two equations are multiplied by -1:

$$\begin{aligned}
&2x_2 - x_3 + x_4 = 5;\\
&- x_1 - x_2 + x_3 + x_5 = 1;\\
&- 2x_1 + x_2 - x_6 = 3.
\end{aligned}$$

In the canonical form of linear programming problems all of the variables included in the system of restrictions should be negative. Assume that $x_1 = x_1^{'} - x_7$, $x_1^{'} \geq 0, x_7 \geq 0$.

Substituting this expression into the objective function and constraints, and writing variables in ascending order of the index, we obtain a linear programming problem presented in the canonical form:

$$\begin{aligned}
&\min z = 2x_1^{'} + x_2 - x_3 - 2x_7 \\
&2x_2 - x_3 = 5; \\
&-x_1^{'} - x_2 + x_3 + x_5 + x_7 = 1; \\
&-2x_1^{'} + x_2 - x_6 + 2x_7 = 3; \\
&x_1^{'} \geq 0; x_i \geq 0; i = \overline{2,7}.
\end{aligned}$$