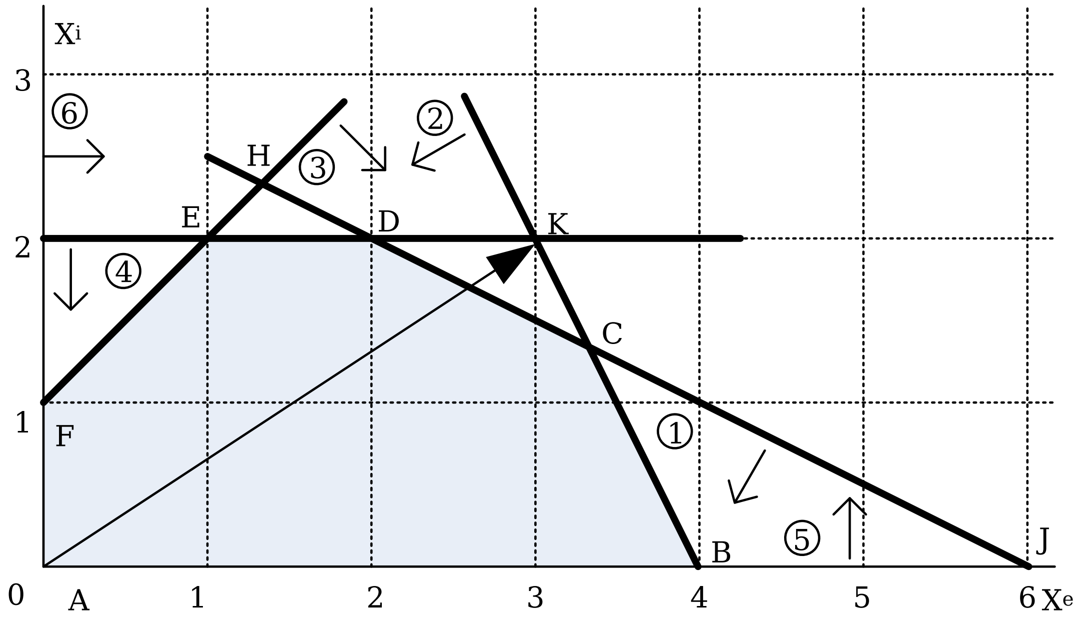

Figure 3: Constraints region of the problem

4 Simplex method

4.1 Idea of the simplex method

According to the fundamental theorem, instead of exploring the infinite set of feasible solutions, it is necessary to consider only a finite number of feasible basic solutions (FBS). Thus, the concept of LPP solution is as follows:

1. Find all FBS.

2. Calculate for each of them corresponding value of OF Z.

3. Compare and determine the best.

But, in general, for large values of n and m number of basic solutions (and hence admissible basis solutions) can be huge (about C_n^m) and the practical implementation of the iteration through all FBS becomes impossible.

These difficulties stem from the fact that this concept is associated with **chaotic brute force** iteration through FBS without taking into account how the new checked FBS changes OF Z and whether it brings us to the desired optimum.

The number of analyzed FBS can be dramatically reduced if we make iterations purposeful achieving monotonic variation of OF, ie each successive FBS was better than the last (or at least not worse).

The main method for solving LPP – simplex method is based on the idea of continuous improvement of solutions. Obviously, for the realization of this idea method should include three main elements:

1. A method for determining the initial FBS.
2. Criterion by which we determine optimality of the solution found, or the need for its further improvement.
3. A rule to advance to the next "best" FBS.

4.2 The scheme of the simplex method

We construct the so-called simplex method for solving $L\ PP\ i\ n$ *canonical form* which has the following schema:

Step 0. ***Construction of the initial FBS***

Find a FBS x^0 of original LPP (this is called the initial FBS). Let this FBS has corresponding

- basis B,
- basis matrix ,
- nonbasic matrix N,
- vector of basic variables $x_B = B^{-1}b$,
- nonbasic variables x_N,
- vector of constraints values $\pi^T = c_B^T B^{-1}$. (in optimal solutions it is valuation of resources)

Step 1. ***Calculation of the components of the relative valuations vector of nonbasic variables.***

$d_{_N}^T = c_B^T B^{-1} N - c_N^T$ or $= \pi_N^T N - c_N^T$.

Step 2. ***The verification of the optimality conditions.***

If $d_N \geq 0$ holds then stop the calculation – current FBS is a solution of the original problem.

Step 3. ***Selecting the nonbasic variables $(x_N)_p$ that gets introduced into the set of basic variables.***

Choose p, for which $(d_N)_p = \max\limits_{j|(d_N)_j<0} |(d_N)_j|$
(Usually to the minimal negative component d_N = the maximum absolute value negative).

Step 4. ***Selecting the basic variable that gets excluded from the set of basic variables.***

Calculate the elements of the *leading column* : $\alpha_p = B^{-1}a_p$.

The condition of admissibility

If $\alpha_p \leq 0$, then stop computing – the objective function is not bounded above.

Otherwise choose q, for which holds $\frac{\beta_q}{\alpha_{qp}} = \min\limits_{\alpha_{ip}>0} \frac{\beta_i}{\alpha_{ip}}$, ie variable $(x_B)_q$ will be excluded from the set of basic variables.

Step 5. ***Replacement operation.***

Construct a basis for a new FBS by replacing the column a_q of current basis to column a_p. Build a new basis matrix and non-basis N. Find new FBS with $x_B = B^{-1}b$. Go to step 1.

Procedure 1-5 is called *iteration of the simplex method* .

If you have a minimum LPP, a sufficient condition for optimality is the condition $d_N \leq 0$.

In practice, number of iterations is $3m$, and the total calculation time km^3 (k – proportionality factor depending on the type of problem).

4.3 Example of using two-stage method

Suppose we have a mathematical model:

$$\min z = 3x_1 + x_2, \quad (10)$$

$$2x_1 + x_2 \geq 2, \quad (11)$$

$$x_1 + 2x_2 \geq 2, \quad (12)$$

$$x_1, x_2 \geq 0 \quad (13)$$

Let us rewrite the problem in the canonical form:

$$\min z = 3x_1 + x_2 + 0s_1 + 0s_2, \tag{14}$$

$$2x_1 + x_2 - 1s_1 \qquad = 2, \tag{15}$$

$$x_1 + 2x_2 \qquad - 1s_2 \;\; = 2, \tag{16}$$

$$x_1, x_2, s_1, s_2 \geq 0. \tag{17}$$

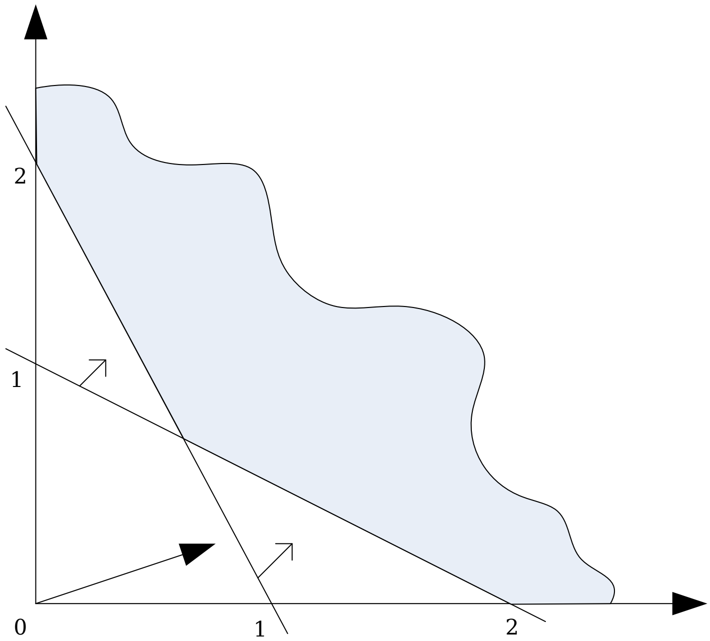

Figure 4

Phase I

1. Introduce artificial variables into the constraints (11) and (12) (because initially these restrictions had the form "$\geq$"). We denote artificial variables by R_1 and R_2, respectively. Then the model (14)-(17) take the following form:

$$\min \; z = 3x_1 + x_2 + 0s_1 + 0s_1 + 0R_1 + 0R_2, \tag{18}$$

$$2x_1 + x_2 - 1s_1 \qquad + R_1 \qquad = 2, \tag{19}$$

$$x_1 + 2x_2 \qquad - 1s_2 \qquad + R_2 = 2, \tag{20}$$

$$x_1, x_2, s_1, s_2, R_1, R_2 \geq 0. \tag{21}$$

2. At the first stage of two-stage method we should minimize the supporting OF $r = \sum_{i=1}^{m} R_i$. In our case: $r = R_1 + R_2$.

We express R_1 and R_2 from equation (19), (20) respectively:

$$R_1 = 2 - 2x_1 - x_2 + 1s_1,$$

$$R_1 = 2 - x_1 - 2x_2 + 1s_2.$$

Substituting these expressions into the objective function r:

$$r = R_1 + R_2 = 4 - 3x_1 - 3x_2 + 1s_1 + 1s_2.$$

Convert it to the following form:

$$r + 3x_1 + 3x_2 - 1s_1 - 1s_2 = 4.$$

3. Build the initial simplex table of two-stage method. Row r is filled according to the expression that was found on the previous step. OF z (18) is transformed to the form:

$$z - 3x_1 - x_2 - 0s_1 - 0s_2 - 0R_1 - 0R_2 = 0.$$

then fill z-row of the table.

Basic variables are variables R_1 and R_2.

Basic variables	x_1	x_2	s_1	s_2	R_1	R_2	Solution	
r(min)	3	3	–1	–1	0	0	4	
z	–3	–1	0	0	0	0	0	
R_1	2	1	–1	0	1	0	2	2/1=2
R_2	1	2	0	–1	0	1	2	2/2=1 (min)

Now we solve the problem using the tabular simplex method by taking row r as the objective function row and on row z will perform the same transformations as over conventional constraints that will allow us after ending the stage I get complete information about the initial FBS of phase II. According to optimality conditions for the minimum problem into the basis introduced a variable, for which there is corresponding positive relative score (positive coefficient of r-row). In our case as x_1 as well as x_2 can be introduced into the basis. We choose variable x_2. Under conditions of admissibility derive from basis variable R_2.

We get the table:

Basic variables	x_1	x_2	s_1	s_2	R_1	R_2	Solution	
r(min)	$\frac{3}{2}$	0	–1	$\frac{1}{2}$	0	$-\frac{3}{2}$	1	
z	$-\frac{5}{2}$	0	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	
R_1	$\frac{3}{2}$	0	–1	$\frac{1}{2}$	1	$-\frac{1}{2}$	1	2/3 (min)
R_2	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	2

Since not all the coefficients of the objective function r are not positive, then continue iteration of the simplex method. Under conditions of optimality we introduce a variable basis x_1 and under conditions of admissibility derive from basis variable R_1. We get the table:

Basic variables	x_1	x_2	s_1	s_2	R_1	R_2	Solution
r(min)	0	0	0	0	–1	–1	0
z	0	0	$-\frac{5}{3}$	$\frac{1}{3}$	$\frac{5}{3}$	$-\frac{1}{3}$	$\frac{8}{3}$
x_1	1	0	$-\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$
x_2	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{3}$

The last table optimality condition for the objective function r holds, that is we found a solution in which this function reaches a minimum. Since the optimal value of function r equals to zero, the initial problem has an acceptable solution - go to stage II.

Phase II

Columns R_1 and R_2 and row r we remove from the current table, and then solve the problem using tabular simplex method by minimizing the objective function z.

Basic variables	x_1	x_2	s_1	s_2	Solution	
z(min)	0	0	$-\frac{5}{3}$	$\frac{1}{3}$	$\frac{8}{3}$	
x_1	1	0	$-\frac{2}{3}$	$\frac{1}{3}$	$\frac{2}{3}$	2
x_2	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	$\frac{2}{3}$	–

According to the optimality conditions we introduce into the basis variable s_2, under conditions of admissibility derive from basis variable x_1. We get the table:

Basic variables	x_1	x_2	s_1	s_2	Solution
z(min)	–1	0	–1	0	2
s_2	3	0	–2	1	2
x_2	2	1	–1	0	2

This table is optimal, because the row z coefficients of the nonbasic variables are non-positive (holds optimality condition for the minimum problem). Problem solved.

Solution: $x_1 = 0$, $x_2 = 2$, $\min z = 2$.

4.4 Special cases occurring in the application of the simplex method

Let LPP given:

$$c^T x \to \max,$$
$$Ax = b,$$
$$x \geq 0.$$

Let x^0 – FBS of constraint system.

Transformed problem corresponding to FBS x^0:

$$c_B^T \beta - d_N^T x_N \to \max,$$
$$x_B + B^{-1} N x_N = \beta,$$
$$x_B \geq 0, x_N \geq 0.$$

Special cases of the use of the simplex method are:

- degeneracy of the solutions;
- unbounded objective function;
- presence of alternative optimum.

4.4.1 The degeneracy of solution

Indication of degeneracy: $\exists i\ \beta_i = 0\ (1 \le i \le m)$ (one or more basic variables take zero value).

Indication of degeneracy by the simplex table corresponding solution:

Basic vars.	$(x_B)_1$	$\dots$	$(x_B)_m$	$(x_N)_1$		$(x_N)_j$		$(x_N)_{n-m}$	Solution
z									
$(x_B)_1$	1								
$\dots$									
$(x_B)_i$		1							0
$\dots$									
$(x_B)_m$			1						

So in LPP shown on Figure 5 A is a degenerate vertex.

Point A correspond to three bases:

$$\{a_{*x_1}, a_{*x_2}, a_{*s_2}, a_{*s_4}\};$$
$$\{a_{*x_1}, a_{*x_2}, a_{*s_1}, a_{*s_4}\};$$
$$\{a_{*x_1}, a_{*x_2}, a_{*s_3}, a_{*s_4}\}.$$

Simplex table for the first of the above bases has the following structure:

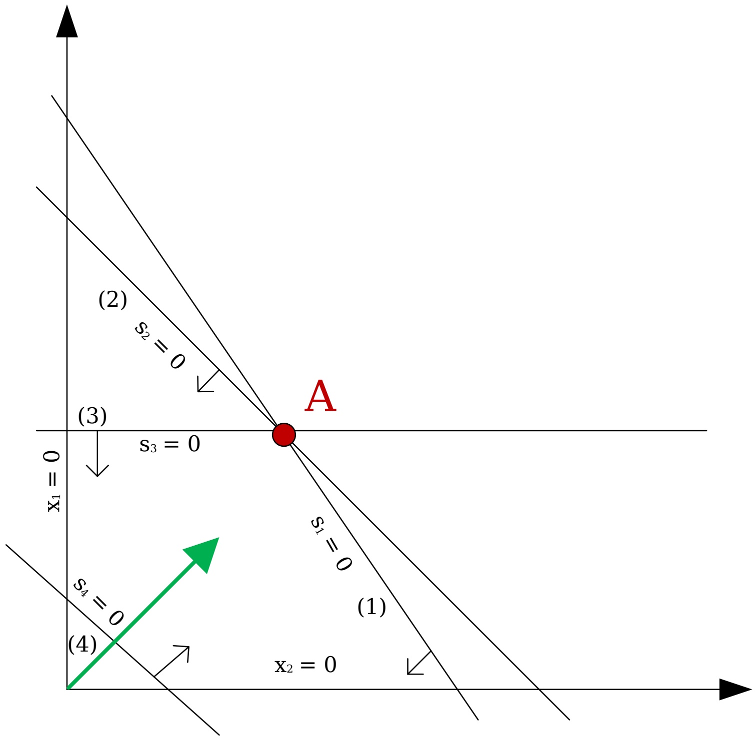

Figure 5

Basic vars.	x_1	x_2	s_2	s_4	s_1	s_3	Solution
z					$+$	$+$	$+$
x_1	1				$+$	$-$	$+$
x_2		1			**0**	$+$	$+$
s_2			1		$-$	$-$	**0**
s_4				1	$+$	**0**	$+$

4.4.2 Unlimited set of feasible solutions

Indication: $\exists\, j \in I_N \;\alpha_{*j} \leq 0$.

Indication of unboundedness of set of feasible solutions by the simplex table:

Basic vars.	$(x_B)_1$	$\dots$	$(x_B)_m$	$(x_N)_1$		$(x_N)_j$		$(x_N)_{n-m}$	Solution
z									
$(x_B)_1$	1					≤ 0			
$\dots$						$\dots$			
$(x_B)_i$		1				≤ 0			
$\dots$						$\dots$			
$(x_B)_m$			1			≤ 0			

4.4.3 Unrestricted objective function

A necessary condition for this is unbounded set of feasible solutions.

Indication (maximum problem): $\exists\, j \in I_N$ ($(d_N)_j < 0 \quad \alpha_{*j} \leq 0$).

Indication of unboundedness of set of feasible solutions by the simplex table:

Basic vars.	$(x_B)_1$	$\dots$	$(x_B)_m$	$(x_N)_1$		$(x_N)_j$		$(x_N)_{n-m}$	Solution
z						< 0			
$(x_B)_1$	1					≤ 0			
$\dots$						$\dots$			
$(x_B)_i$		1				≤ 0			
$\dots$						$\dots$			
$(x_B)_m$			1			≤ 0			

LPP shown on Figure 6 has unlimited from above objective function.

FBS A is represented by the simplex table:

Basic vars.	x_1	x_2	s_3	s_1	s_2	Solution
z				$+$	$-$	$+$
x_1	1			$-$	$-$	$+$
x_2		1		$+$	$-$	$+$
s_3			1	$+$	0	$+$

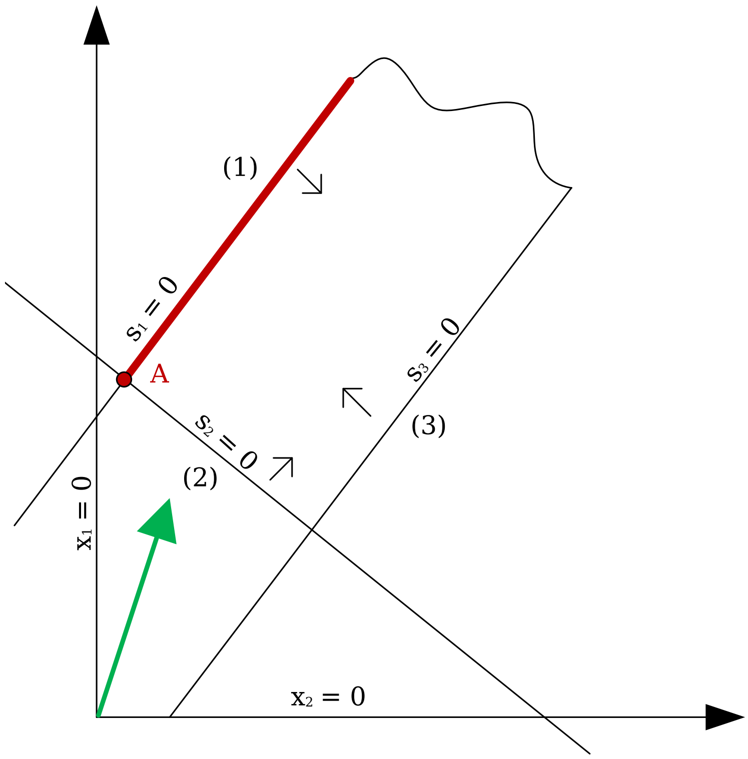

Figure 6: ***Note***: lines (1) and (3) are parallel.

In the case of the minimization problem indication is as follows: $\exists\ j \in I_N$ ($(d_N)_j > 0$ $\alpha_{*j} \leq 0$).

4.4.4 Alternate optimum

Indication of alternate optimum: $d_N \geq 0$, $\exists\ j$ $(d_N)_j = 0$ (one or more nonbasic variables have zero relative score).

Indication of alternate optimum by the simplex table corresponding solution:

Basic vars.	$(x_B)_1$	$\dots$	$(x_B)_m$	$(x_N)_1$		$(x_N)_j$		$(x_N)_{n-m}$	Solution
z				>0		0		>0	
$(x_B)_1$	1								
$\dots$									
$(x_B)_i$		1							
$\dots$									
$(x_B)_m$			1						

There are three such cases:

1. alternate optimum – (infinite) bounded set;
2. alternate optimum – (infinite) unbounded set;
3. when there is an indication of the alternate optimum only one point is optimal.