# 5 The transportation problem

Transportation problem one of the most common problems of mathematical programming (usually linear). In general terms, it can be represented as follows: need to find a plan for the delivery of goods from suppliers to consumers, so that the cost of transportation (or the total range, or the amount of transport work in tonne-kilometers) was the lowest. Consequently, the problem reduces to the most rational attachment of producers to consumers of products (and vice versa). In its simplest form, when one type of product is distributed and consumers do not care from what of the suppliers to receive it, the problem is formulated as follows.

There are a number of production centres $A_1, A_2, \ldots, A_m$ with the volume of production per unit of time (month, quarter) equal respectively $a_1, a_2, \ldots, a_m$, and consumption centres $B_1, B_2, \ldots, B_n$, consuming over the same amount of time, respectively $b_1, b_2, \ldots, b_n$ of product. If we solve closed (balanced) problem, the amount of production volumes at all m-points is equal to the sum of suppliers consumption volumes for all n points of recipient:

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

There are also known costs of transportation of the product from each supplier to each recipient – we denote these values $c_{ij}$. As the unknown quantities are the volumes of the product carried from each production centre in to each consumption centre, respectively denoted $x_{ij}$.

Then the most efficient attachment of suppliers to consumers will be that at which the total cost of transportation will be the lowest:

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \to \min$$

In addition, each user gets the right amount of product

$$\sum_{i=1}^{m} x_{ij} = b_i,$$

and each supplier ships all products produced by them

$$\sum_{j=1}^{n} x_{ij} = a_j,$$

As in all such cases, it also stipulates nonnegative variables: supply from some centre of production to a particular centre of consumption may be zero, but negative (to move in the opposite direction) can not be.

Since it is assumed that the cost of transportation is growing in proportion to their volume, then we have a linear programming problem – one of the problems of resource allocation.

Unbalanced (open) transportation problem transformed to the form shown above artificially: we introduce into the model so-called dummy producer or consumer that balance the demand and consumption.

## 5.1 Method of potentials

***Method of potentials*** – one of the most commonly used methods for solving TLPP. This method is an implementation of the simplex method in terms of the transportation problem.

Next TLPP

$$\min z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \tag{22}$$

$$\sum_{j=1}^{n} x_{ij} = a_i, i = \overline{1, m} \tag{23}$$

$$\sum_{i=1}^{m} x_{ij} = b_j, j = \overline{1, n} \tag{24}$$

$$x_{ij} \geq 0, i = \overline{1, m}, j = \overline{1, n} \tag{25}$$

will be specified by the table, which we call ***transportation table***:

| | | | | |
|---|---|---|---|---|
| $c_{11}$ $x_{11}$ | $c_{12}$ $x_{12}$ | ... | $c_{1n}$ $x_{1n}$ | $a_1$ |
| $c_{21}$ $x_{21}$ | $c_{22}$ $x_{22}$ | ... | $c_{2n}$ $x_{2n}$ | $a_2$ |
| ... | ... | ... | ... | ... |
| $c_{m1}$ $x_{m1}$ | $c_{m2}$ $x_{m2}$ | ... | $c_{mn}$ $x_{mn}$ | $a_m$ |
| $b_1$ | $b_2$ | ... | $b_n$ | $d$ |

Number of rows equal to the number of manufacturers $m$ (constraint (23)); number of columns – the number of consumers $n$ (constraint (24)); each cell of the table corresponds to a particular pair of manufacturer $i$ – consumer $j$; each pair $ij$ correspond to the cost of $c_{ij}$ transportation a unit of production and volume of transportations (number of products) $x_{ij}$ on the route $i$, $j$.

LPP solution using the simplex method starts with some admissible basic solution (FBS). In the method of potentials, the following ways of finding the initial FBS:

- the northwest corner method;
- method of least cost;
- Vogel's method.

### 5.1.1 The method of least-cost

Calculations are made as follows.

**Step 1.** Select the variable, which corresponds to the lowest cost in the whole table, and give it the greatest possible value. (If there are several variables, then any one of them can be taken.)

**Step 2.** Struck off the appropriate column or row (since this constraint of the problem is satisfied). If the constraints on the column and row are satisfied simultaneously, as in the method of the north-western corner, delete either a column or a row.

**Step 3.** Calculate the new value of demand or production volume for undeleted rows or columns.

**Step 4.** After finding the transportation volume on the route specified in step 1, we are dealing with a new problem, in which the total number of producers and consumers 1 less than the initial one.

The process is then repeated with the greatest possible value of that variable, which corresponds to the minimum cost of undeleted (this value can be equal to zero). The procedure is completed when there is one row and one column.

**Example**

For next TLPP

| | | | | |
|---|---|---|---|---|
| 5 | 3 | 6 | 7 | 10 |
| 4 | 5 | 8 | 1 | 12 |
| 8 | 2 | 9 | 10 | 13 |
| 14 | 8 | 7 | 6 | |

we construct a solution using least cost method.

Iteration 1

| | | | | |
|---|---|---|---|---|
| 5 | 3 | 6 | 7 | 10 |
| 4 | 5 | 8 | 1 **6** | ~~12~~ 6 |
| 8 | 2 | 9 | 10 | 13 |
| 14 | 8 | 7 | ~~6~~ – | |

Iteration 2

| | | | | | | |
|---|---|---|---|---|---|---|
| 5 | 3 | 6 | 7 | 10 | | |
| 4 | 5 | 8 | 1 **6** | ~~12~~ | 6 | |
| 8 | 2 **8** | 9 | 10 | ~~13~~ | 5 | |
| 14 | ~~8~~ | 7 | ~~6~~ | | | |
| | – | | – | | | |

Iteration 3

| | | | | | | |
|---|---|---|---|---|---|---|
| 5 | 3 | 6 | 7 | 10 | | |
| 4 **6** | 5 | 8 | 1 **6** | ~~12~~ | ~~6~~ | – |
| 8 | 2 **8** | 9 | 10 | ~~13~~ | 5 | |
| ~~14~~ | ~~8~~ | 7 | ~~6~~ | | | |
| 8 | – | | – | | | |

Iteration 4

| | | | | | | |
|---|---|---|---|---|---|---|
| 5 **8** | 3 | 6 | 7 | ~~10~~ | 2 | |
| 4 **6** | 5 | 8 | 1 **6** | ~~12~~ | ~~6~~ | – |
| 8 | 2 **8** | 9 | 10 | ~~13~~ | 5 | |
| ~~14~~ | ~~8~~ | 7 | ~~6~~ | | | |
| ~~8~~ | – | | – | | | |
| – | | | | | | |

Iteration 5

| | | | | | | |
|---|---|---|---|---|---|---|
| 5 **8** | 3 | 6 **2** | 7 | ~~10~~ | ~~2~~ | – |
| 4 **6** | 5 | 8 | 1 **6** | ~~12~~ | ~~6~~ | – |
| 8 | 2 **8** | 9 | 10 | ~~13~~ | 5 | |
| ~~14~~ | ~~8~~ | ~~7~~ | ~~6~~ | | | |
| ~~8~~ | – | 5 | – | | | |
| – | | | | | | |

Iteration 6

| | | | | | | |
|---|---|---|---|---|---|---|
| 5 **8** | 3 | 6 **2** | 7 | ~~10~~ | ~~2~~ | – |
| 4 **6** | 5 | 8 | 1 **6** | ~~12~~ | ~~6~~ | – |
| 8 | 2 **8** | 9 **5** | 10 | ~~13~~ | ~~5~~ | – |
| ~~14~~ | ~~8~~ | ~~7~~ | ~~6~~ | | | |
| ~~8~~ | – | ~~5~~ | – | | | |
| – | | – | | | | |

As the result of method's application $6 = m + n - 1 = 3 + 4 - 1$ cells of the transportation table has been filled. Thus, the solution obtained by the method of least cost: $x_{11} = 8,\ x_{13} = 2,\ \ x_{21} = 6,\ x_{24} = 6,\ x_{32} = 8,\ x_{33} = 5\ \ x_{ij} = 0,$

$$z = 40 + 12 + 24 + 6 + 16 + 45 = 143.$$

Then it is necessary to:

- find potentials
- find the relative valuations of nonbasic variables
- check optimality criteria

***Finding the relative valuations of nonbasic variables***

$$u_1 = 0$$

$$d_{ij} = u_i + v_j - c_{ij},\ ij \in I_N$$

(for all occupied cells of the table)

| | $v_1 = 5$ | | $v_3 = 6$ | |
|---|---|---|---|---|
| $u_1 = 0$ | 5 **8** | 3 | 6 **2** | 7 |
| | 4 **6** | 5 | 8 | 1 **6** |
| | 8 | 2 **8** | 9 **5** | 10 |

| | $v_1 = 5$ | $v_2 = -1$ | $v_3 = 6$ | $v_4 = 2$ |
|---|---|---|---|---|
| $u_1 = 0$ | 5 **8** | 3 | 6 **2** | 7 |
| $u_2 = -1$ | 4 **6** | 5 | 8 | 1 **6** |
| $u_3 = 3$ | 8 | 2 **8** | 9 **5** | 10 |

***Finding the relative valuations of nonbasic variables***

$$d_{ij} = u_i + v_j - c_{ij},\ \ ij \in I_N$$

(for all UNoccupied table cells)

| | $v_1 = 5$ | $v_2 = -1$ | $v_3 = 6$ | $v_4 = 2$ |
|---|---|---|---|---|
| $u_1 = 0$ | 5 **8** | 3 **-4** | 6 **2** | 7 **-5** |
| $u_2 = -1$ | 4 **6** | 5 **-7** | 8 **-3** | 1 **6** |
| $u_3 = 3$ | 8 **0** | 2 **8** | 9 **5** | 10 **-5** |

Since all $d_{ij} \leq 0$, the current solution is optimal!

**Solution:**
$x_{11} = 8,\ x_{13} = 2,\ \ x_{21} = 6,\ x_{24} = 6,\ x_{32} = 8,\ \ x_{33} = 5\ \ \ x_{ij} = 0,$

$$z = 143.$$