3 Graphical method of solving LPP

Linear programming problem (LPP) can be solved graphically, if the problem has no more than two variables. Let us find the solution of the problem comprising determining the maximum value of the function under the conditions

$$z = c_1x_1 + c_2x_2 \tag{1}$$

$$a_{i1}x_1 + a_{i2}x_2 \leq b_i, i = 1, \dots, m, \tag{2}$$

$$x_1, x_2 \geq 0. \tag{3}$$

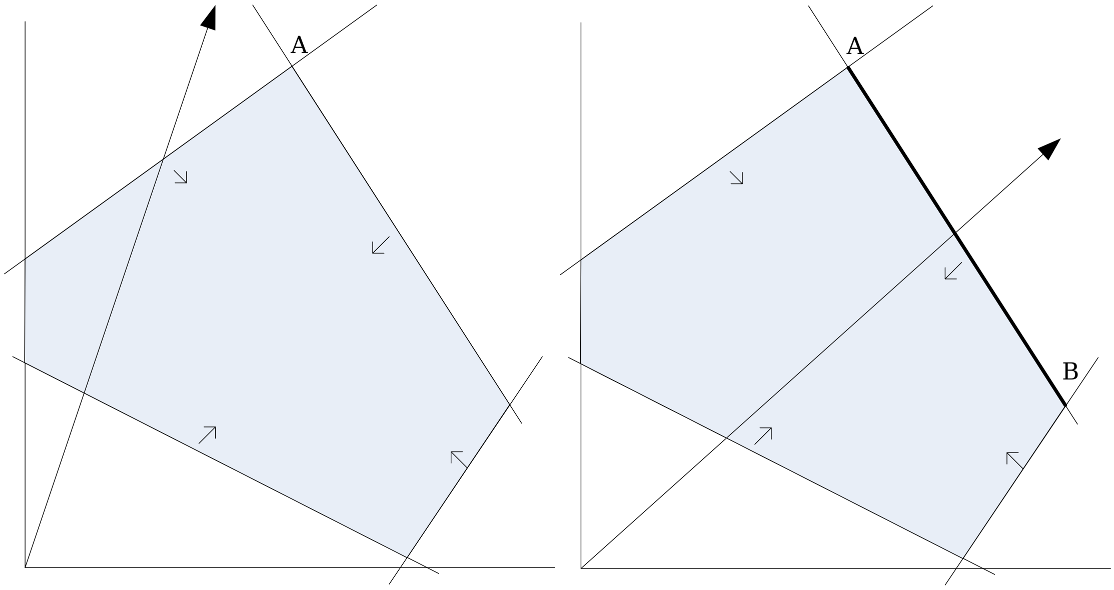
Each of inequalities of the constraint system (2)-(3) geometrically defines a half-plane, respectively, with the boundary lines $a_{i1}x_1 + a_{i2}x_2 = b_i (i = 1, \dots, m)$, $x_1 = 0$, $x_2 = 0$. In that case, if the system of inequalities (2)-(3) is consistent the range of its solutions is the set of points that belong to all of the half-planes. In general, the region of admissible solutions of the problem (1)-(3) is a convex polygon. Side of the polygon lie on the lines whose equations are obtained from the original constraint system by replacing the signs of inequalities with the exact equality signs.

Thus, the original LPP is to find a point of solutions of the polygon in which the objective function z takes the maximum value. This point exists when the polygon of solutions is non-empty and the objective function is bounded from above. Under specified conditions in one of the vertices of the polygon of solutions the objective function takes the maximum value. To determine this vertex we construct line level $c_1x_1 + c_2x_2 = h$ (where h is a constant), passing through the polygon of solutions and will move in the direction of the vector (normal) $N = (c_1, c_2)$ until it passes through the last point in common with its solution polygon. Coordinates of a specified point determine the best solution to this problem.

Feasible region (FR) of inequality system (1)-(3) may be empty, a single point, segment, beam, or an unbounded convex polygon region.

On Figures 1a-2b are some cases that can be encountered during the problem solution finding (1)-(3).

Figure 1a describes the case where the objective function takes the maximum value at a single point. Figure 1b shows that the objective function has maximum value at any point of the segment AB (direct objective function is parallel to the constraint represented by the line AB). In such cases we say that the problem has an alternative optimum.



(a) (b)

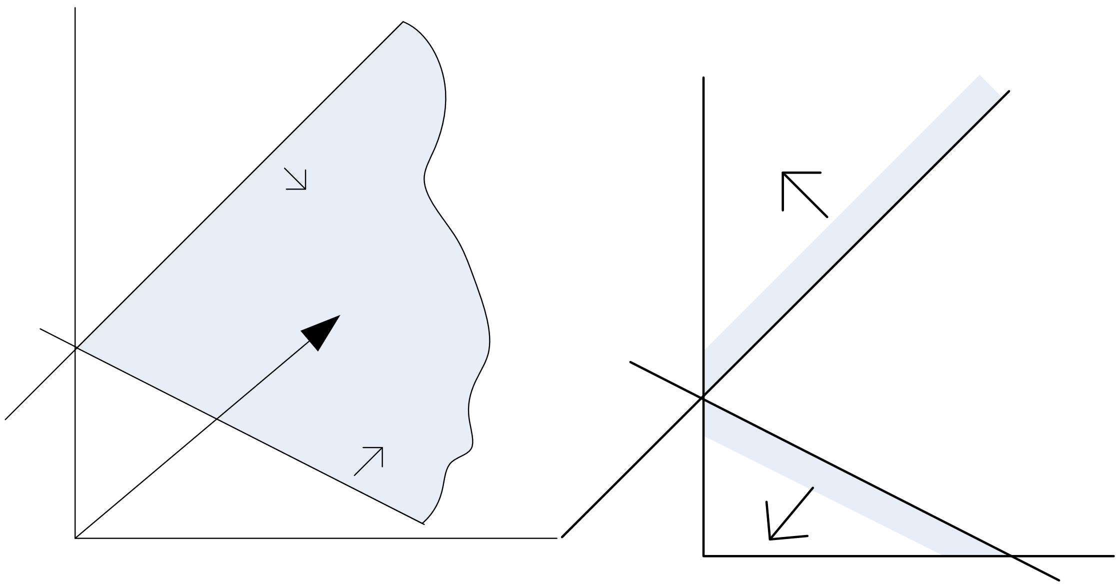
Figure 1: Examples of problems with the consistent system and limited objective function

So, if the feasible region is a convex polygon (limited region), the maximum and minimum of the linear function z is reached at least one of the vertices of the polygon. If the extreme value of z is reached at two vertices (the case of alternative optimum), the same extreme value is reached in any point on the segment joining the two vertices.

In the case of an unbounded region maximum (minimum) of the function z either does not exist if z is unbounded from above (below, or is reached at least at one of the vertices of the area. Figure 2a shows the case where the objective function is not bounded from above on the set of feasible solutions, and Figure 2b shows the case where the system of constraints is inconsistent.

Thus, the geometric method for solving the LPP for a maximum includes the following steps:

1. The construction of the boundary lines, the equations of which are obtained by replacing in the constraints (2) and (3) inequality signs with the exact



(a) (b)

Figure 2: Examples of special cases

equality signs.

2. Finding the half-planes defined by each of the inequality constraints of the problem. To determine on which side of the boundary line the half-plane corresponding to a given inequality is located, it suffices to verify any point (the easiest way is to take point (0,0)). If after substituting its coordinates in the left-hand side the inequality is satisfied, the halfplane faces the point of measurement, if the inequality is not satisfied, then the corresponding halfplane faces the opposite direction. The direction of the half-plane is indicated by sign "$\rightarrow$" or shading. (Inequalities $x_1 \geq 0$ and $x_2 \geq 0$ also correspond to the half-planes).

3. Finding solutions polygon.

4. Construction of the vector $N = (c_1, c_2)$.

5. Construction of the line $c_1x_1 + c_2x_2 = h$ that passes through the solutions polygon.

6. Moving the line $c_1x_1 + c_2x_2 = h$ to the direction of vector N. whereby

finding point(s) in which the objective function takes the maximum value, or determining unbounded function from above on the set of solutions.

7. Determination of the coordinates of optimum point of the function and calculation of the objective function value at this point.

Note that finding the minimum value of a linear function for a given set of constraints is different from the location of its maximum value at the same constraints only in that line-level $c_1x_1 + c_2x_2 = h$ moves not along the vector $N = (c_1, c_2)$, but in the opposite direction.

3.1 Example solution of LPP using graphical method

$$\max z = 3x_E + 2x_i \tag{4}$$
$$1x_E + 2x_i \leq 6 \tag{5}$$
$$2x_E + 1x_i \leq 8 \tag{6}$$
$$-1x_E + 1x_i \leq 1 \tag{7}$$
$$1x_i \leq 2 \tag{8}$$
$$x_E, x_i \geq 0 \tag{9}$$

Feasible region of the problem is the polygon ABCDEF (Figure 3). The optimal solution to the problem - point C (the closest point of intersection of constraints to the point of the objective function vector vertex). Values x_E and x_i at this point are determined by solving the system of two equations $1x_E+2x_i = 6$, $2x_E+1x_i = 8$. Solving it, we get: $x_E = \frac{10}{3}$, $x_i = \frac{4}{3}$. Revenue z received in this case, will be $\frac{38}{3}$ (thousand dollars).

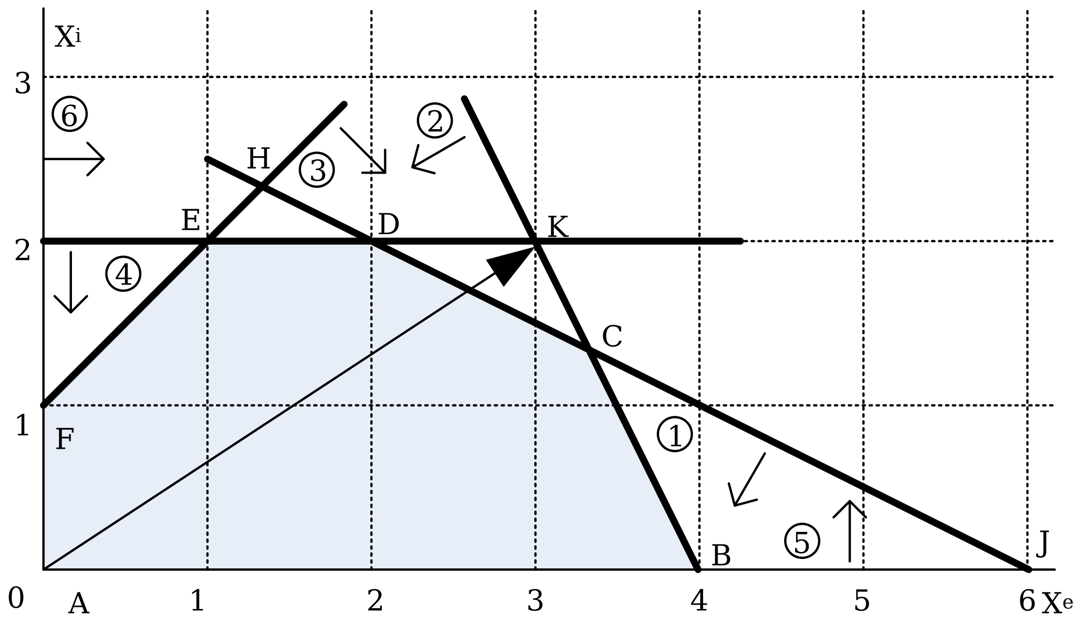


Figure 3: Constraints region of the problem

4 Simplex method

4.1 Idea of the simplex method

According to the fundamental theorem, instead of exploring the infinite set of feasible solutions, it is necessary to consider only a finite number of feasible basic solutions (FBS). Thus, the concept of LPP solution is as follows:

1. Find all FBS.
2. Calculate for each of them corresponding value of OF Z.
3. Compare and determine the best.

But, in general, for large values of n and m number of basic solutions (and hence admissible basis solutions) can be huge (about C_n^m) and the practical implementation of the iteration through all FBS becomes impossible.

These difficulties stem from the fact that this concept is associated with **chaotic brute force** iteration through FBS without taking into account how the new checked FBS changes OF Z and whether it brings us to the desired optimum.